GAUGE EQUIVALENCE AND INVERSE SCATTERING FOR LONG-RANGE MAGNETIC POTENTIALS

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ABSTRACT. For Schrödinger operators with long-range magnetic vector potentials and short range electric scalar potentials in an exterior domain Ω in $\mathbf{R}^n$ with $n \geq 2$, we show that there is a one-to-one correspondence between the gauge equivalent classes of Hamiltonians and those of S-matrices, if Ω is exterior to a bounded convex obstacle.

1. INTRODUCTION

Let $\Omega = \mathbf{R}^n \setminus \overline{\mathcal{O}}$, where $\mathcal{O}$ is a bounded domain with smooth boundary, called an *obstacle*. We consider a Schrödinger operator

$$(1.1) \qquad H = (-\nabla_x - A(x))^2 + V(x)$$

in Ω with Dirichlet boundary condition, where $A(x) = (A_1, ..., A_n) \in C^\infty(\overline{\Omega})$ is the magnetic vector potential and $V(x) \in C^\infty(\overline{\Omega})$ is the electric scalar potential. We assume that $A(x), V(x)$ are real-valued, and satisfy the following conditions: $V(x)$ is a short-range potential, i.e.

$$(1.2) \qquad |\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-1-\varepsilon_0-|\alpha|}, \quad \forall \alpha,$$

where $\varepsilon_0 > 0$ and $\langle x \rangle = (1 + |x|^2)^{1/2}$, $A(x)$ is a long-range potential, i.e.

$$(1.3) \qquad |\partial_x^\alpha A(x)| \leq C_\alpha \langle x \rangle^{-1-|\alpha|}, \quad \forall \alpha.$$

Throughout the paper, C's denote various constants independent of the variable x (and other parameters). We also assume that $A(x)$ satisfies the *transversal gauge condition*

$$(1.4) \qquad |\partial_x^\alpha (x \cdot A(x))| \leq C_\alpha \langle x \rangle^{-1-|\alpha|}, \quad \forall \alpha.$$

It is well-known that the scattering operator S exists under the conditions (1.2), (1.3), (1.4) (cf. [LoTh87]). The properties of S related to the Aharonov-Bohm effect and the scattering problem were discussed in [AB59], [Rui83], [RoYa02], and others.

In this paper, we shall consider a slightly more restrictive class of long-range magnetic potentials than (1.3). Namely, in addition to the conditions (1.2), (1.3) and (1.4), we assume for some $R > 0$

$$(1.5) \qquad A(x) = A_0(x) + A_1(x) \ \text{ for } \ |x| > R,$$

where $A_0(x) \in C^\infty(\mathbf{R}^n \setminus \{0\})$, homogeneous of degree -1, and satisfies

$$(1.6) \qquad x \cdot A_0(x) = 0.$$

We also assume that $A_1(x)$ is a short-range potential decaying like (1.2). The condition (1.6) implies that $A_1(x)$ satisfies (1.4).

In order to define a gauge transformation group, we need to introduce two spaces of functions. Let $\mathcal{L}_{\infty,0}$ be the set of real-valued functions $L(x) \in C^\infty(\overline{\Omega})$ satisfying

$$(1.7) \qquad |\partial_x^\alpha L(x)| \leq C_\alpha \langle x \rangle^{-\epsilon_0 - |\alpha|}, \quad \forall \alpha,$$

Let $\mathcal{H}_0$ be the set of $C^\infty(\mathbf{R}^n \setminus \{0\})$-functions, homogeneous of degree 0. We sometimes write $\psi(x) \in \mathcal{H}_0$ as $\psi(\hat{x}) \in C^\infty(S^{n-1})$, where $\hat{x} = x/|x|$. When $n = 2$, we also write $\varphi \in \mathcal{H}_0$ as $\varphi(\theta)$, where θ is the polar angle, i.e. $\hat{x} = (\cos\theta, \sin\theta)$.

Note that when $\varphi(x) \in \mathcal{H}_0$ the vector potential $\nabla_x \varphi(x)$ satisfies the transversal gauge condition

$$(1.8) \qquad x \cdot \nabla_x \varphi(x) = 0.$$

Indeed, let $r = |x|$. Then $x \cdot \nabla_x \varphi(x) = r\frac{\partial}{\partial r}\varphi(\hat{x}) = 0$ since $\varphi(\hat{x})$ is independent of r.

Definition 1.1. Denote by $\mathbf{G}(\overline{\Omega})$ the group of $C^\infty(\overline{\Omega})$ functions having the following properties: $|g(x)| = 1$ in $\overline{\Omega}$ and there exists a constant $R > 1$ such that for $|x| > R$

$$(1.9) \qquad g(x) = \begin{cases} e^{i(\psi(\hat{x})+L(x))}, & \psi \in \mathcal{H}_0, \quad L \in \mathcal{L}_{\infty,0}, \quad \text{when } n \geq 3, \\ e^{i(m\theta+\varphi(\theta)+L(x))}, & \varphi \in \mathcal{H}_0, \quad L \in \mathcal{L}_{\infty,0}, \quad \text{when } n = 2, \end{cases}$$

where m is an integer (which may depend on $g(x)$), and

$$(1.10) \qquad \int_0^{2\pi} \varphi(\theta)d\theta = 0.$$

We shall call $\mathbf{G}(\overline{\Omega})$ the *gauge group* and the transformation $u(x) \to g(x)u(x)$ the *gauge transformation*.

If $u(x)$ is the solution to the Schrödinger equation $Hu = \lambda u$, $v(x) = g(x)u(x)$ satisfies $H'v = \lambda v$, where

$$H' = \left(-i\frac{\partial}{\partial x} - A'\right)^2 + V',$$

with $V'(x) = V(x)$, and

$$(1.11) \qquad A'(x) = A(x) - ig^{-1}\frac{\partial g}{\partial x}.$$

Magnetic potentials $A'(x)$ and $A(x)$ satisfying (1.11) are called *gauge equivalent*. Let $\widehat{S}(\lambda)$ and $\widehat{S}'(\lambda)$ be the scattering matrices corresponding to H and H', respectively. Then (see, for example, [Ya06])

$$(1.12) \qquad \widehat{S}'(\lambda) = e^{i\psi(D_x)}\widehat{S}(\lambda)e^{-i\psi(-D_x)}$$

when $n \geq 3$, and

$$(1.13) \qquad \widehat{S}'(\lambda) = e^{im\theta(D_x)+i\varphi(D_x)}\widehat{S}(\lambda)e^{-im\theta(-D_x)-i\varphi(-D_x)}$$

when $n = 2$. Scattering matrices $\widehat{S}(\lambda)$ and $\widehat{S}'(\lambda)$ satisfying (1.12) or (1.13) are said to be *gauge equivalent*. Therefore, if the magnetic vector potentials are gauge equivalent, so are the scattering matrices. A natural question is the validity of the converse assertion. This is an inverse problem. Since the domain Ω could be

multiply-connected, our task shoud be not only the recovery of the magnetic field but the recovery of the gauge equivalence classes of the electro-magnetic potentials.

In our previous work [EIO], we have dealt with the case $n = 2$ (and $\mathcal{H}_0 \ni \varphi(\theta) = 0$), and proven that if the scattering matrices coincide, the Schrödinger operators are gauge equivalent under the equal flux condition. Moreover, outside the convex obstacles, the equal flux condition is satisfied. Therefore, outside the convex obstacle, there is a one-to-one correspondence between the gauge equivalence classes of Schrödinger operators and those of S-matrices. The aim of this article is to extend this result to the case $n \geq 2$ including $\mathcal{H}_0$.

At the first sight, this problem seems to be of the topological nature, and the case with dimension $n \geq 3$ looks simpler than the case $n = 2$. However, as it will be shown, results for $n = 2$ and $n = 3$ are more or less the same and they are determined by the principal part $A_0(x)$ of the long-range magnetic potential.

2. Inverse scattering problem

The following theorem is one of our main results.

Theorem 2.1. *Let $n \geq 2$ and $H^{(j)} = \left(-i\frac{\partial}{\partial x} + H^{(j)}(x)\right)^2 + V^{(j)}$ be two Schrödinger operators in $\Omega^{(j)}, j = 1, 2$, with Dirichlet boundary condition on $\partial\Omega^{(j)}$. Here $A^{(j)} = A_0^{(j)} + A_1^{(j)}, j = 1, 2$, satisfy (1.5), (1.6), and $V^{(j)}(x)$ satisfy (1.2). Moreover, when $n = 2$ we assume that*

$$(2.1) \qquad |V^{(1)}(x) - V^{(2)}(x)| \leq C_N \langle x \rangle^{-N}, \quad \forall N,$$

$$(2.2) \qquad |A_1^{(1)}(x) - A_1^{(2)}(x)| \leq C_N \langle x \rangle^{-N}, \quad \forall N.$$

Suppose $\widehat{S}^{(1)}(\lambda) = \widehat{S}^{(2)}(\lambda)$ for all $\lambda > 0$ and suppose there exists $R > 0$ such that $A_0^{(1)}(x) = A_0^{(2)}(x)$ for $|x| > R$. Then $\Omega^{(1)} = \Omega^{(2)}$, $V^{(1)} = V^{(2)}$ in Ω and $A^{(1)}$ and $A^{(2)}$ are gauge equivalent in Ω.

Proof: Let $B_R = \{x : |x| < R\}$ be a ball that contains the complements to Ω_1 and Ω_2. Let $\{x_0 + s\omega, s \in \mathbf{R}, \omega \in S^{n-1}\}$ be a line not intersecting B_R. It was shown in [Nic00], [We02], [BaWe07] that $\hat{S}^{(1)}(\lambda) = \hat{S}^{(2)}(\lambda)$ for all $\lambda > 0$ implies

$$(2.3) \qquad \exp\left(i \int_{-\infty}^{\infty} A^{(1)}(x_0 + s\omega) \cdot \omega ds\right) = \exp\left(i \int_{-\infty}^{\infty} A^{(2)}(x_0 + s\omega) \cdot \omega ds\right),$$

$$(2.4) \qquad \int_{-\infty}^{\infty} V^{(1)}(x_0 + s\omega)ds = \int_{-\infty}^{\infty} V^{(2)}(x_0 + s\omega)ds.$$

Note that (1.6) implies that integrals in (2.3) converge. Since we assume that $A_0^{(1)} = A_0^{(2)}$ we get from (2.3) that

$$(2.5) \qquad \exp\left(i \int_{-\infty}^{\infty} (A_1^{(1)} - A_1^{(2)})(x_0 + s\omega) \cdot \omega dt\right) = 1.$$

Therefore $\int_{-\infty}^{\infty} (A_1^{(1)} - A_1^{(2)})(x_0 + s\omega) \cdot \omega dt = 2\pi m$, where m is integer. Since this integral tends to zero when $|x_0| \to \infty$ we get that $m = 0$.

Consider first the case $n \geq 3$. Let Π be an arbitrary two-dimensional plane in the exterior of B. Then it follows from (2.4), (2.5) (cf. [Nic00], [Es03], [BaWe07])

that $V^{(1)} = V^{(2)}$ on Π and $B_1^{(1)}|_\Pi = B_1^{(2)}|_\Pi$, where $B_1^{(i)} = dA_1^{(i)}$ and $B_1^{(1)}|_\Pi$ is the restriction of the two-form $B_1^{(i)}$ to Π. Since Π is an arbitrary plane we get that

$$(2.6) \qquad V^{(1)} = V^{(2)}, \quad B_1^{(1)} = B_1^{(2)} \quad \text{when} \ \ |x| \geq R.$$

Therefore there exists $L_1 \in \mathcal{L}_{\infty,0}$ such that

$$(2.7) \qquad A_1^{(2)} - A_1^{(1)} = \nabla_x L_1 \ \ \text{for} \ \ |x| \geq R,$$

since $\mathbf{R}^n \setminus B_R$ is a simply-connected domain when $n \geq 3$.

In the case $n = 2$ we get (2.6), (2.7) under the assumption (2.1), (2.2) (cf. [He99], [Nic00], [EIO]).

Extend arbitrarily $L_1(x)$ from $|x| \geq R$ to $\mathbf{R}^n, n \geq 2$. Make the gauge transformation of $H^{(2)}$ with the gauge $g_1(x) = e^{iL_1(x)}$. Then $H^{(1)}$ and $H^{(3)} = g_1^{-1} H^{(2)} g_1$ coincide for $|x| \geq R$. Note that $\widehat{S}^{(3)}(\lambda) = \widehat{S}^{(2)}(\lambda)$ since $\nabla_x L_1$ is a short-range potential. Now as in the proof of Theorem 5.7 in [EIO] we want to show that $\widehat{S}^{(1)}(\lambda) = \widehat{S}^{(2)}(\lambda)$ and $H^{(3)} = H^{(1)}$ for $|x| \geq R$ implies that the Dirichlet-to-Neumann operators $\Lambda^{(1)}(\lambda)$ and $\Lambda^{(3)}(\lambda)$ for $H^{(1)}$ and $H^{(3)}$ respectively coincide on $C = \{|x| = R\}$ for any $\lambda > 0$. Theorem 5.7 of [EIO] is not trivial in the long-range case. Fortunately, the results of §2, §3, 4.1, 5.1 of [EIO] hold for any $n \geq 2$ when conditions (1.2), (1.3), (1.4) are satisfied. In particular, we have the spatial asymptotics of the resolvent and also that of the eigenoperator, whose kernel is the distorted plane wave (cf. Lemma 3.9 and Theorems 4.4 and 4.5 in [EIO]). Next using the BC-method (cf. [Be97], [KuLa00], [Es06], [Es07]) we get that $\Lambda^{(1)}(\lambda) = \Lambda^{(2)}(\lambda)$ for any $\lambda > 0$ implies that $\Omega^{(1)} = \Omega^{(2)}, V^{(1)} = V^{(2)}$ in Ω and $A^{(2)} - A^{(1)} = ig^{-1}\nabla_x g$ in Ω (cf. Theorem 5.8 in [EIO]). $\qquad\square$

Theorem 2.1 proves the main result under the additional assumption $A_0^{(1)} = A_0^{(2)}$. In the next two sections we show that this condition is necessary when $H^{(1)}$ and $H^{(2)}$ are gauge equivalent and it follows from the equality of the scattering matrices $\widehat{S}^{(1)}(\lambda) = \widehat{S}^{(2)}(\lambda)$ in the case when $\Omega^{(1)} = \Omega^{(2)}$, and it is an exterior to a convex obstacle.

3. The two-dimensional case

We first consider the 2-dimensional case. Let us start with the following fact.

Lemma 3.1. *Let* $A_0(x) \in C^\infty(\mathbf{R}^2 \setminus \{0\})$, *and* $A_0(x)$ *is homogeneous of degree* -1. *Suppose* $A_0(x) \cdot x = 0$. *Then there exists* $a_0(x) \in \mathcal{H}_0$ *such that*

$$(3.1) \qquad A_0(x) = \alpha \frac{(-x_2, x_1)}{|x|^2} + \nabla_x a_0(x),$$

$$(3.2) \qquad \alpha = \frac{1}{2\pi} \int_{|x|=R} A_0(x) \cdot dx, \quad \int_0^{2\pi} a_0(\cos\theta, \sin\theta)d\theta = 0.$$

Proof: Let $A_0(x) = (A_{01}(x), A_{02}(x))$. Then $A_0(x) \cdot x = 0$ implies $x_1 A_{01} + x_2 A_{02} = 0$, and

$$\frac{A_{02}}{x_1} = \frac{A_{01}}{-x_2} = \frac{a(x)}{|x|^2},$$

where $a(x)$ is homogeneous of degree zero. Therefore

$$(3.3) \qquad A_{01} = -\frac{x_2}{|x|^2}a(x), \quad A_{02} = \frac{x_1}{|x|^2}a(x).$$

Since $a(x) = a(x/|x|)$ we have in polar coordinates $x_1 = r\cos\theta, x_2 = r\sin\theta$ that $a(x/|x|) = a(\cos\theta, \sin\theta)$. Let $\hat{a}(\theta) = a(\cos\theta, \sin\theta)$, i.e. $\hat{a}(\theta)$ is 2π-periodic function of θ, and let

$$(3.4) \qquad \alpha = \frac{1}{2\pi}\int_0^{2\pi}\hat{a}(\theta)d\theta.$$

Note that α in (3.2), (3.4) is the same. Moreover, α can be defined as the limit: $\alpha = \lim_{R\to\infty}\frac{1}{2\pi}\int_{|x|=R}A(x)\cdot dx$.

Let $b(\theta) = \hat{a}(\theta) - \alpha$. Then $\int_0^{2\pi}b(\theta)d\theta = 0$. Therefore $b(\theta)$ can be represented in the form

$$b(\theta) = \frac{\partial}{\partial\theta}\hat{a}_0(\theta),$$

where $\hat{a}_0(\theta)$ is 2π-periodic and

$$\int_0^{2\pi}\hat{a}_0(\theta)d\theta = 0.$$

Let $a_0(x/|x|) \in \mathcal{H}_0$ be such that $a_0(\cos\theta, \sin\theta) = \hat{a}_0(\theta), \ \theta = \tan^{-1}(x_1/x_2)$. We have

$$\frac{\partial a_0}{\partial x_1} = \frac{\partial\hat{a}_0(\theta)}{\partial\theta}\frac{\partial\theta}{\partial x_1} = \frac{x_2}{|x|^2}\frac{\partial\hat{a}_0(\theta)}{\partial\theta},$$

$$\frac{\partial a_0}{\partial x_2} = \frac{\partial\hat{a}_0(\theta)}{\partial\theta}\frac{\partial\theta}{\partial x_2} = -\frac{x_1}{|x|^2}\frac{\partial\hat{a}_0(\theta)}{\partial\theta}.$$

Therefore we have proven

$$(3.5) \qquad \begin{aligned} A_0(x) &= \frac{(-x_2, x_1)}{|x|^2}\hat{a}(\theta) = \frac{(-x_2, x_1)}{|x|^2}\left(\alpha + \frac{\partial\hat{a}_0(\theta)}{\partial\theta}\right) \\ &= \alpha\frac{(-x_2, x_1)}{|x|^2} + \nabla_x a_0(x). \qquad\qquad \square \end{aligned}$$

It follows from Lemma 3.1 that when $n = 2$ any magnetic potential of the form (1.5), (1.6) is gauge equivalent in Ω to the Aharonov-Bohm type potential

$$(3.6) \qquad A_\alpha = \alpha\frac{(-x_2, x_1)}{|x|^2} + A_1'(x),$$

where A_1' is a short-range potential.

It was shown in [Ya06], [RoYa02] that the kernel of scattering matrix $\widehat{S}_\alpha(\lambda)$ corresponding to (3.6) has the form

$$(3.7) \qquad S_\alpha(\theta, \theta') = S_\alpha(\theta - \theta') + S_\alpha'(\theta, \theta'),$$

where

$$(3.8) \qquad S_\alpha(\theta) = \cos(\alpha\pi)\delta(\theta) + i\frac{\sin(\alpha\pi)}{\pi}\text{p.v.}\frac{e^{i[\alpha]\theta}}{1 - e^{i\theta}},$$

$$(3.9) \qquad |S_\alpha'(\theta, \theta')| \le C|\theta - \theta'|^{-\delta}, \ 0 < \delta < 1,$$

$[\alpha]$ is the integer part of α. (In [Ya06], [RoYa02] the case $\Omega = \mathbf{R}^2$ was considered. The extension to the case of domains with obstacles was shown in [EIO]).

It follows from Lemma 3.1 and (1.13) that the kernel of the scattering matrix $\widehat{S}(\lambda)$ corresponding to $A(x) = A_0 + A_1$ has the form (cf. [RoYa02], [Ya06])

$$(3.10) \qquad S(\theta, \theta') = e^{i\hat{a}_0(\theta) - i\hat{a}_0(\pi + \theta')}(S_\alpha(\theta - \theta') + S'_\alpha(\theta, \theta')).$$

The next theorem shows that in the case of a convex obstacle $\hat{S}^{(1)}(\lambda) = \hat{S}^{(2)}(\lambda)$ implies that $A_0^{(1)} = A_0^{(2)}$ if α is not an integer.

Theorem 3.2. *Let* $H^{(j)} = \left(-i\frac{\partial}{\partial x} + A^{(j)}\right)^2 + V^{(j)}$ *be two Schrödinger operators in the same domain* $\Omega \subset \mathbf{R}^2$, *where* Ω *is the exterior to a convex obstacle. Suppose* $V^{(j)}, j = 1, 2$, *are short-range potentials and*

$$A^{(j)} = \alpha_j \frac{(-x_2, x_1)}{|x|^2} + \nabla_x \varphi_j(x) + A_1^{(j)}, \quad j = 1, 2,$$

where $\varphi_j(x) \in \mathcal{H}_0$, $\int_{S^1} \varphi_j(\hat{x}) d\sigma = 0$, $A_1^{(j)}$ *are short-range potentials,* $j = 1, 2$. *Suppose* $\widehat{S}^{(1)}(\lambda) = \widehat{S}^{(2)}(\lambda)$ *for all* $\lambda > 0$ *and* α_1 *is not integer. Suppose also that* (2.1), (2.2) *hold. Then*

$$(3.11) \qquad \alpha_1 = \alpha_2, \quad \varphi_1(x) = \varphi_2(x),$$

$V^{(1)} = V^{(2)}$ *in* Ω *and there exists* $L_1(x) \in \mathcal{L}_{\infty,0}$ *such that* $A_1^{(2)} - A_1^{(1)} = \nabla_x L_1$.

Proof: As has been proved in [Nic00], [We02], $\widehat{S}^{(1)}(\lambda) = \widehat{S}^{(2)}(\lambda)$ for all $\lambda > 0$ implies (2.3), (2.4) for all lines $x = x_0 + s\omega$ not intersecting Ω. Let $\omega = (\cos\theta, \sin\theta)$, $x_1 = x_{10} + s\cos\theta$, $x_2 = x_{20} + s\sin\theta$, $x_0 \cdot \omega = 0$. Then

$$(3.12) \qquad \int_{-\infty}^{\infty} \alpha \frac{(-x_2, x_1)}{|x|^2} \cdot \omega ds = \alpha \int_{-\infty}^{\infty} \frac{|x_0|}{|x_0|^2 + s^2} ds = \alpha\pi,$$

$$(3.13) \qquad \int_{-\infty}^{\infty} \frac{\partial \varphi(x_0 + s\omega)}{\partial x} \cdot \omega ds = \int_{-\infty}^{\infty} \frac{d}{ds}\varphi(x_0 + s\omega) ds$$

$$= \varphi(x_0 + s\omega)|_{-\infty}^{\infty} = \varphi(\frac{1}{|s|}x_0 + \frac{s}{|s|}\omega)|_{-\infty}^{\infty} = \varphi(\omega) - \varphi(-\omega),$$

where we used that φ is homogeneous of degree 0. Note that $\int_{-\infty}^{\infty} A_1^{(j)}(x_0 + s\omega) \cdot \omega ds \to 0$ when $|x_0| \to \infty$, $x_0 \cdot \omega = 0$. Therefore taking the limit in (2.3) when $|x_0| \to \infty$ we get

$$(3.14) \qquad (\alpha_2 - \alpha_1)\pi + \varphi(\omega) - \varphi(-\omega) = 2\pi m,$$

for some integer m, where $\varphi(\omega) = \varphi_2(\omega) - \varphi_1(\omega)$. It follows from (3.14) that $\varphi(\omega) - \varphi(-\omega)$ does not depend on ω. Since $\varphi(\omega) - \varphi(-\omega)$ is odd, there exists ω_0 such that $\varphi(\omega_0) - \varphi(-\omega_0) = 0$. Therefore

$$(3.15) \qquad \varphi(\omega) - \varphi(-\omega) = 0, \quad \forall \omega \in S^1,$$

$$(3.16) \qquad \alpha_2 - \alpha_1 = 2m.$$

It follows from (3.14) and (2.3) that (2.5) holds. Therefore using (2.4), (2.5), (2.1) and (2.2), we get as in the proof of Theorem 2.1 (cf. [Nic00], [BaWe07], [Es03]) that $V^{(1)} = V^{(2)}$ in Ω and (2.7) holds in Ω. As in the proof of Theorem

2.1, let $H^{(3)} = g_1^{-1} H^{(2)} g_1$ where $g_1 = e^{iL_1}$. Let $\widehat{S}^{(3)}(\lambda)$ is the scattering matrix corresponding to $H^{(3)}$. Note also that $H^{(3)}$ has the form

$$H^{(3)} = \left(-i\frac{\partial}{\partial x} + \alpha_2 \frac{(-x_2, x_1)}{|x|^2} + \nabla_x \varphi_2 + A_1^{(1)}\right)^2 + V^{(1)}(x).$$

Let $g(x) = e^{i2m\theta + i\varphi(\hat{x})}$ where $\hat{x} = (\cos\theta, \sin\theta)$. Then $g^{-1} H^{(3)} g = H^{(1)}$. Therefore (cf. (1.13))

$$(3.17) \qquad S^{(3)}(\theta, \theta') = e^{i2m(\theta - \theta') + i\varphi(\omega) - i\varphi(-\omega)} S^{(1)}(\theta, \theta'),$$

where (cf. (3.7), (3.8))

$$(3.18) \qquad S^{(1)}(\theta, \theta') = (S_{\alpha_1}(\theta - \theta') + S'_{\alpha_1}(\theta, \theta')) e^{i\varphi_1(\omega) - i\varphi_1(-\omega)},$$

$$(3.19) \qquad |S'_{\alpha_1}(\theta, \theta')| \leq C|\theta - \theta'|^{-\delta}, \quad 0 \leq \delta < 1.$$

Since $\widehat{S}^{(1)} = \widehat{S}^{(3)}$ we get

$$(3.20) \qquad (e^{i2m(\theta - \theta') + i(\varphi(\omega) - \varphi(-\omega))} - 1) S^{(1)}(\theta, \theta') = 0.$$

Denote $\varphi_0(\theta) = \varphi(\cos\theta, \sin\theta)$. Then $\varphi(-\cos\theta, -\sin\theta) = \varphi(\cos(\theta + \pi), \sin(\theta + \pi)) = \varphi_0(\theta + \pi)$. We have $\varphi_0(\theta) - \varphi_0(\theta + \pi) = 0$, $\forall\theta$. Using the Taylor formula we get

$$(3.21) \qquad \begin{aligned} \varphi_0(\theta) - \varphi_0(\theta' + \pi) &= \varphi_0(\theta') + \varphi_0'(\theta')(\theta - \theta') + O((\theta - \theta')^2) - \varphi_0(\theta' + \pi) \\ &= \varphi_0'(\theta')(\theta - \theta') + O((\theta - \theta')^2). \end{aligned}$$

Since α_1 is not an integer we have for arbitrary θ_0':

$$(3.22) \qquad |S^{(1)}(\theta, \theta')| = |S_{\alpha_1}(\theta - \theta') + S'_{\alpha_1}(\theta, \theta')| \geq C|\theta - \theta_0'|^{-1}$$

for all $0 < \theta - \theta' < \varepsilon$, ε is small. Take θ_0' such that $\varphi_0'(\theta_0') = 0$. Then using (3.21) and assuming $m \neq 0$ we get

$$(3.23) \qquad |e^{i2m(\theta - \theta_0') + i(\varphi_0(\theta_0') - \varphi_0(\theta_0' + \pi))} - 1| \geq C|\theta - \theta_0'|.$$

Therefore the left-hand side of (3.20) is nonzero for $0 < \theta - \theta_0' < \varepsilon$. This contradiction proves that $2m = \alpha_2 - \alpha_1 = 0$.

Assuming that $\varphi_0(\theta)$ is not zero we can find θ_0'' such that $\varphi'(\theta_0'') \neq 0$. Then taking into account that $m = 0$ we get

$$(3.24) \qquad |e^{i(\varphi_0(\theta) - \varphi_0(\theta_0'' + \pi))} - 1| \geq C|\theta - \theta_0''| \quad \text{for} \quad |\theta - \theta_0''| \text{ small}.$$

Again (3.24) and (3.22) contradict (3.20). Therefore $\varphi_0'(\theta) = 0$ for all θ and hence $\varphi_0(\theta) = \varphi_2(\theta) - \varphi_1(\theta) = 0$, $\forall\theta$. $\qquad\square$

The following theorem shows that the condition $A_0^{(1)} = A_0^{(2)}$ is necessary if one assumes that $\widehat{S}^{(1)}(\lambda) = \widehat{S}^{(2)}(\lambda)$ and $H^{(1)}$ and $H^{(2)}$ are gauge equivalent.

Theorem 3.3. *Let Ω be an arbitrary domain with obstacles. Let $H^{(j)} = (-i\frac{\partial}{\partial x} + A^{(j)}(x))^2 + V^{(j)}(x)$ be the Schrödinger operators in Ω as in Theorem 2.1. Suppose $\widehat{S}^{(1)}(\lambda) = \widehat{S}^{(2)}(\lambda)$ and suppose that $H^{(1)}$ and $H^{(2)}$ are gauge equivalent, i.e. there exists $g(x) \in \mathbf{G}(\overline{\Omega})$ such that $H^{(1)} = g^{-1} H^{(2)} g$. Suppose also that*

$$\alpha_1 = \lim_{R \to \infty} \frac{1}{2\pi} \int_{|x|=R} A^{(1)}(x) \cdot dx$$

is not an integer. Then $A_0^{(1)}(x) = A_0^{(2)}(x)$.

Proof: Denote $H^{(3)} = g^{-1}H^{(2)}g$. Then $H^{(3)} = H^{(1)}$ and therefore $\widehat{S}^{(3)}(\lambda) = \widehat{S}^{(1)}(\lambda)$. It follows from (1.13) that

$$S^{(3)}(\theta, \theta') = e^{im(\theta-\theta')+i(\varphi_0(\theta)-\varphi_0(\theta+\pi))} S^{(2)}(\theta, \theta').$$

Since $S^{(2)}(\theta, \theta') = S^{(1)}(\theta, \theta')$ we get the same equation as in (3.20):

$$(3.25) \qquad (e^{i2m(\theta-\theta')+i(\varphi_0(\theta)-\varphi_0(-\theta+\pi))} - 1)S^{(1)}(\theta, \theta') = 0,$$

i.e. we are in the same situation as in the proof of Theorem 3.2. Therefore $m = 0, \varphi_0(\theta) = 0$. Hence $A_0^{(2)} = A_0^{(1)}$. $\qquad\square$

On the basis of Theorem 3.2 we can formulate our ultimate result in 2-dimensions.

Theorem 3.4. *Let Ω be the exterior of a convex obstacle. Let $H^{(j)}, j = 1, 2$, be the same as in Theorem 3.2. Let $\widehat{S}^{(j)}(\lambda)$ be the scattering matrices corresponding to $H^{(j)}, j = 1, 2$. If $\widehat{S}^{(1)}(\lambda)$ and $\widehat{S}^{(2)}(\lambda)$ are gauge equivalent for all $\lambda > 0$, then $V^{(1)} = V^{(2)}$ and $A^{(1)}$ and $A^{(2)}$ are gauge equivalent.*

Proof: Since $\widehat{S}^{(1)}(\lambda)$ and $\widehat{S}^{(2)}(\lambda)$ are gauge equivalent there exists $g(x) = e^{im\theta+i\varphi_0(\theta)}$ such that (1.13) holds. Let $H^{(3)} = g^{-1}H^{(2)}g$. Then $\widehat{S}^{(1)}(\lambda) = \widehat{S}^{(3)}(\lambda)$ where $\widehat{S}^{(3)}(\lambda)$ is the scattering matrix for $H^{(3)}$. Also we have that $A_0^{(3)}$ is gauge equivalent to $A_0^{(2)}$, $A_1^{(3)} = A_1^{(2)}$, $V^{(3)} = V^{(2)}$. Since $\widehat{S}^{(3)}(\lambda) = \widehat{S}^{(1)}(\lambda)$ we get applying Theorem 3.2 that $A_0^{(3)} = A_0^{(1)}$, $V^{(3)} = V^{(1)}$ and $A_1^{(3)} - A_1^{(1)} = \nabla_x L_1$. Therefore $A^{(2)} = A_0^{(2)} + A_1^{(2)}$ is gauge equivalent to $A^{(1)} = A_0^{(1)} + A_1^{(1)}$ and $V^{(1)} = V^{(2)}$. $\qquad\square$

We end this section with a conjecture that Theorem 3.4 holds for arbitrary domain with obstacles.

4. THE CASE $n \geq 3$

The following theorem is the analog of Theorem 3.2 for $n \geq 3$.

Theorem 4.1. *Let Ω be the exterior of a convex obstacles in $\mathbf{R}^n, n \geq 3$. Let $H^{(j)} = (-i\frac{\partial}{\partial x} - A^{(j)})^2 + V^{(j)}(x)$ be two Schrödinger operators as in Theorem 2.1. Suppose $\operatorname{curl} A_0^{(1)} \neq 0$, and that $\widehat{S}^{(1)}(\lambda) = \widehat{S}^{(2)}(\lambda)$ for all $\lambda > 0$. Then $V^{(1)} = V^{(2)}$, $A_1^{(2)} - A_1^{(1)} = \nabla_x L_1$ with $L_1 \in \mathcal{L}_{\infty,0}$ and $A_0^{(1)} = A_0^{(2)}$.*

Proof: Consider (2.3), (2.4). Let $A_0 = A_0^{(2)} - A_0^{(1)}$, $A_1 = A_1^{(2)} - A_1^{(1)}$. It follows from (2.3) $\int_{-\infty}^{\infty}(A_0 + A_1)(x_0 + s\omega) \cdot \omega ds = 2\pi m$ where $m \in \mathbf{Z}$. As in Theorem 2.1 take any two-dimensional plane Π in Ω. The proof in [Es03], pp 990-991, give that $B|_\Pi = 0$ even when $m \neq 0$. Here $B = B_0 + B_1, B_0 = dA_0, B_1 = dA_1, B$ is the restriction of 2-form B to Π. Since Π is arbitrary we have as in Theorem 2.1 that $B_0 + B_1 = 0$, $V^{(1)} = V^{(2)}$ in Ω. Note that $B_0(x) = b(x)/|x|^2$ where $b(x)$ is homogeneous of degree zero and $|B_1(x)| \leq C\langle x \rangle^{-2-\varepsilon_0}$. Multiplying $B_0 + B_1 = 0$ by $|x|^2$ and taking the limit when $|x| \to \infty$ we get that $b(x) = 0$. Therefore $B_0 = 0$ and $B_1 = 0$. Since Ω is a simply connected domain, we have $A_1 = \nabla_x L_1$, $A_0 = \nabla_x \varphi$, where $L_1 \in \mathcal{L}_{\infty,0}$ and $\varphi(x) \in \mathcal{H}_0$. Since $\int_{-\infty}^{\infty} \nabla_x L_1 \cdot \omega ds = 0$, we get from (2.3) that $\int_{-\infty}^{\infty} A_0 \cdot \omega ds = 2\pi m$.

Therefore as in (3.13)

$$(4.1) \quad 2\pi m = \int_{-\infty}^{\infty} \frac{\partial \varphi}{\partial x}(x_0 + s\omega) \cdot \omega ds = \int_{-\infty}^{\infty} \frac{d}{ds}\varphi(x_0 + s\omega)ds = \varphi(\omega) - \varphi(-\omega).$$

Changing ω to $-\omega$ we get $m = 0$, i.e. $\varphi(\omega) - \varphi(-\omega) = 0, \forall \omega \in S^{n-1}$. Let $g(x) = e^{i\varphi(\omega)+iL_1(x)}$ and let $H^{(3)} = g^{-1}H^{(2)}g$. Then $H^{(3)} = H^{(1)}$ and therefore $\widehat{S}^{(3)} = \widehat{S}^{(1)}$. By (1.12) we have $\widehat{S}^{(3)}(\omega,\omega') = \widehat{S}^{(2)}(\omega,\omega')e^{i\varphi(\omega)-i\varphi(-\omega')}$. Since $\widehat{S}^{(1)} = \widehat{S}^{(2)}$ we have

$$(4.2) \qquad\qquad (e^{i(\varphi(\omega)-\varphi(-\omega'))} - 1)S^{(1)}(\omega,\omega') = 0.$$

As in the proof of Theorem 3.2, we have by the Taylor formula

$$(4.3) \qquad\qquad e^{i(\varphi(\omega)-\varphi(-\omega'))} = e^{i(\varphi_x(\omega')\cdot(\omega-\omega')+O(|\omega-\omega'|^2))},$$

since $\varphi(\omega') - \varphi(-\omega') = 0$. Therefore if ω_0' is such that $\varphi_x(\omega_0') \neq 0$ then

$$(4.4) \qquad\qquad |e^{i(\varphi(\omega)-\varphi(-\omega'))} - 1| \geq C|\omega - \omega'|,$$

when $|\omega-\omega'|$ is small, ω' is close to ω_0' and $(\omega-\omega')\cdot\varphi_x(\omega_0') \neq 0$. Since curl $A_0^{(1)} \neq 0$, we get that $\nabla_x I(x,\omega) \neq 0$, where $I(x,\omega) = \int_{-\infty}^{\infty} A_0^{(1)}(x + s\omega) \cdot \omega ds$. Then it follows from [Ya06] that $S^{(1)}(\omega,\omega')$ contains a principal value integral and therefore $S^{(1)}(\omega,\omega') \neq 0$ when (4.4) holds. This contradicts (4.2). Therefore we must have $\varphi = 0$, i.e. $A_0^{(1)} = A_0^{(2)}$. $\qquad\qquad\square$

The proof of the following Theorems are the same as that of Theorems 3.3 and 3.4 with the reference to Theorem 3.2 replaced by the reference to Theorem 4.1.

Theorem 4.2. *Suppose $\Omega_1 = \Omega_2 = \Omega$ is an arbitrary domain with obstacles in $\mathbf{R}^n, n \geq 3$, and $H^{(1)}$ and $H^{(2)}$ are the same as in Theorem 4.1. Suppose curl $A_0^{(1)} \not\equiv 0$. If $\widehat{S}^{(1)}(\lambda) = \widehat{S}^{(2)}(\lambda)$ and if $H^{(1)}$ and $H^{(2)}$ are gauge equivalent, then $A_0^{(1)} = A_0^{(2)}$.*

Theorem 4.3. *Suppose $\Omega_1 = \Omega_2 = \Omega \subset \mathbf{R}^n, n \geq 3$, where Ω is the exterior of a convex obstacle. Let $H^{(j)} = \left(-\frac{\partial}{\partial x} + A^{(j)}\right)^2 + V^{(j)}$, $\widehat{S}^{(j)}(\lambda)$, $j = 1, 2$, be the same as in Theorem 4.1. If $\widehat{S}^{(1)}(\lambda)$ and $\widehat{S}^{(2)}(\lambda)$ are gauge equivalent for all $\lambda > 0$, then $H^{(1)}$ and $H^{(2)}$ are gauge equivalent.*

It follows from Theorem 4.2 that the assertion: $\widehat{S}^{(1)}(\lambda) = \widehat{S}^{(2)}(\lambda)$ implies $A_0^{(1)} = A_0^{(2)}$ is necessary when $H^{(1)}$ and $H^{(2)}$ are gauge equivalent. This leads us to a conjecture that Theorem 4.1 and consequently Theorem 4.3 are true for any obstacle.

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